M.Sc. Sem-II

Complex Analysis — PAPER-V

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Th. 2 Sufficient condition for $f(z)$ to be analytic

The function $w = f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if

1.) u, v are differentiable in D and $u_x = v_y$, $u_y = -v_x$

2. The partial derivative u_x, u_y, v_x, and v_y are all ~~condition~~ continuous in D.

Proof: We have $w = u + iv$ $\therefore \delta w = \delta u + i\delta v$

$$\delta u = u(x+\delta x, y+\delta y) - u(x,y) \quad -(1)$$

$$= [u(x+\delta x, y+\delta y) - u(x, y+\delta y)] + [u(x, y+\delta y) - u(x,y)] \quad -2$$

The above step may be noted we have subtracted and added $u(x, y+\delta y)$

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Again by mean value Theorem we know that if $f(x)$ is continuous in $a \leq x \leq b$ and differentiable in $a < x < b$ then

$$f(a+h) - f(a) = hf'(a+\theta h)$$
where $0 < \theta < 1$.

Applying the result of above theorem in both the breackets in (2) we get

$$\delta u = \delta x, u_x(x+\theta\delta x, y+\delta y) + \delta y \cdot u_y(x, y+\theta\delta y)$$
— (3)

Where $0 < \theta < 1$ & $0 < \theta < 1$
Again u_x and u_y are given to be continuous

$$|u_x(x+\theta\delta x, y+\delta y) - u_x(x,y)| < \epsilon$$

$$|u_y(x, y+\theta'\delta y) - u_y(x,y)| < \eta$$

Now choosing $\epsilon_1 < \epsilon$ and $\eta_1 < \eta$

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We have from above

$$U_x(x + \theta\delta x, y + \delta y) - U_x(x, y) = \epsilon_1$$

$$U_y(x, y + \theta\delta y) - U_y(x, y) = \eta_1$$

Hence from (3) by the help of above relation we get

$$\delta u = (U_x (x, y) + \epsilon_1)$$
$$\delta x + \delta u_y (x, y) + \eta_1)\,\delta y \quad - (4)$$

Proceeding exactly as above, we get

$$\delta v = V_x (x, y) + \epsilon_2)\,\delta x +$$
$$(V_y(x, y) + \eta_2)\,\delta y \quad - (5)$$

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Putting the values of δu and δv from (4) and (5) in (1) and writing $u_x(x,y)$ simply at u_x and similarly for other we get.

$$\delta w = [(u_x + \epsilon_1)\delta x + (u_y + \eta_1)\delta y$$
$$+ i[v_x + \epsilon_2)\delta x + (v_y + \eta_2)\delta y$$

or

$$\delta w = (u_x + i v_x)\delta x + (u_y + i v_y)\delta y$$
$$+ (\epsilon_1 + i\epsilon_2)\delta x + (\eta_1 + i\eta_2)\delta y \quad \text{— (5)}$$

Now by Cauchy Riemman equation $u_x = v_y$ and $u_y = -v_x = i^2 v_x$ and choosing $\epsilon_3 = \epsilon_1 + i\epsilon_2$ and $\eta_3 = \eta_1 + i\eta_2$

Hence (6) can be written as

$$\delta w = (u_x + i v_x)\delta x + i(i v_x + u_x)\delta y$$
$$+ \epsilon_3 \cdot \delta x + \eta_3 \delta y$$
$$= (u_x + i v_x)(\delta x + i\delta y) + \epsilon_3 \delta x + \eta_3 \delta y \quad \text{— (7)}$$

Dividing throughOut by

$\delta z = \delta x + i\delta y$ we get

$$\frac{\delta w}{\delta z} = u_x + iv_x + \frac{e_3 \delta x + n_3 \delta y}{\delta x + i\delta y}$$

Taking limit when $\delta z \to 0$ so that $\delta x \to 0$, $\delta y \to 0$, $e_3 \to 0$, $n_3 \to 0$ we get

$$f'(z) = \frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z}$$

$$= u_x + iv_x .$$

Since u_x, v_x exist and are unique therefore from above we conclude that $f'(x)$ exist. Therefore $f(z)$ is analytic at an arbitary point z of D and hence it is analytic at the Domain D.

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